

# K-string tensions at finite temperature and integrable models

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**Michele Caselle, Pietro Giudice, Ferdinando Gliozzi, Paolo Grinza, Stefano Lottini**

*Dipartimento di Fisica Teorica, Università di Torino and  
INFN, Sezione di Torino  
via P.Giuria 1, I-10125 Torino, Italy  
caselle,giudice,gliozzi,grinza,lottini@to.infn.it*

**ABSTRACT:** It has recently been pointed out that simple scaling properties of Polyakov correlation functions of gauge systems in the confining phase suggest that the ratios of k-string tensions in the low temperature region is constant up to terms of order  $T^3$ . Here we argue that, at least in a three-dimensional  $\mathbb{Z}_4$  gauge model, the above ratios are constant in the whole confining phase. This result is obtained by combining numerical experiments with known exact results on the mass spectrum of an integrable two-dimensional spin model describing the infrared behaviour of the gauge system near the deconfining transition.

**KEYWORDS:** Lattice Gauge Field Theories, Confinement, Duality, Integrable Models.

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## 1. Introduction

In most confining gauge theories, besides the fundamental string (of tension  $\sigma$ ) which is formed between a pair of static sources in the fundamental representation  $f$ , there is the freedom of taking the sources in any representation  $\mathcal{R}$ .

If, for instance, the gauge group is  $SU(N)$  there are infinitely many irreducible representations at our disposal. However, as the sources are pulled apart, no matter what representation is chosen, the asymptotically stable string tension  $\sigma_{\mathcal{R}}$  depends only on the  $N$ -ality  $k$  of  $\mathcal{R}$ , i.e. on the number (modulo  $N$ ) of copies of the fundamental representation needed to build  $\mathcal{R}$  by tensor product, because all representations with the same  $k$  can be transformed into each other by the emission of a proper number of soft gluons. As a consequence the heavier strings decay into the string of smallest string tension  $\sigma_k$ . The corresponding string is referred to as a  $k$ -string. This kind of confining object can be defined whenever the gauge group admits more than one non trivial irreducible representation.

Much work has been done in the study of  $k$ -string tensions in the continuum [1]–[8] as well as on the lattice [9]–[16].

In a previous work [17], some of us have argued from simple scaling properties of suitable Polyakov loop correlators that these string tensions have the following low temperature asymptotic expansion

$$\sigma_k(T) = \sigma_k - c \frac{\pi}{6} T^2 + \mathcal{O}(T^3) ; \quad c = (d-2) \frac{\sigma_k}{\sigma} , \quad (1.1)$$

where  $c$  is the central charge of the underlying 2D conformal field theory describing the IR behaviour of the  $k$ -string. As a consequence, their ratios are expected to be constant up to  $T^3$  terms:

$$\frac{\sigma_k(T)}{\sigma(T)} = \frac{\sigma_k}{\sigma} + \mathcal{O}(T^3) . \quad (1.2)$$

The low temperature data presented in support of this expectation were taken from Monte Carlo simulations on a particular system, namely a (2+1)-dimensional  $\mathbb{Z}_4$  gauge model, which is the simplest exhibiting more than just the fundamental string.

The main conjecture we want to verify in this work is that  $\sigma_k(T)/\sigma(T)$ , at least in that  $\mathbb{Z}_4$  gauge system, is in fact independent of the temperature in the *whole* of the confining regime. To check this idea, a handy fact comes useful, namely that, as the system approaches the deconfinement transition, and the string picture begins fading, another approach is made available by the Svetitsky-Yaffe (SY) conjecture [18], which allows to reformulate the system in a totally different perspective, based on a two-dimensional integrable theory in which, however, the near- $T_c$  counterpart of the low-temperature result cited above can be nicely found.

It turns out that the deconfinement transition of the 3D  $\mathbb{Z}_4$  gauge model is second order and, according to the SY conjecture, belongs to the same universality class of the 2D symmetric Ashkin-Teller (AT) model. As a matter of fact, such a model possesses a whole line of critical points along which the critical exponents vary continuously. The SY conjecture tells us that if a (2+1)-dimensional gauge model with center  $\mathbb{Z}_4$  displays a second-order transition, then its universality class is associated to a suitable point of the critical line of the 2D AT model. For instance, it has been argued [19] that the critical (2+1)D  $SU(4)$  gauge theory belongs to the universality class of a special point of the AT model, known as the four-state Potts model. More generally, the class of models with gauge group  $\mathbb{Z}_4$  depends on two coupling constants  $\alpha$  and  $\beta$ , and the universality class of the deconfining point  $P$  varies with the ratio  $\alpha/\beta$ .

The two-dimensional AT model can be seen in the continuum limit as a bosonic conformal field theory plus a massive perturbation (i. e. a Sine-Gordon theory) driving the system away from the critical line. Thus, a map between (a neighbourhood of) the AT critical line and the Sine-Gordon phase space is provided.

This theory is integrable, and the masses of its lightest physical states (first soliton and first breather mode, of masses  $M$  and  $M_1$ ) correspond to the tensions  $\sigma(T)$  and  $\sigma_2(T)$  near  $T_c$ , whose ratio, in this context, can be analytically evaluated and turns out to be

$$\lim_{T \rightarrow T_c} \frac{\sigma_2(T)}{\sigma(T)} = \frac{M_1}{M} = 2 \sin \frac{\pi}{2} (2\nu - 1) , \quad (1.3)$$

where  $\nu$  is the thermal exponent in two dimensions.

As a consequence, on the gauge side, we have two different ways to verify the conjecture. One is to directly estimate the ratio  $M_1/M$  by measuring the Polyakov-Polyakov correlators in the two non-trivial representations of  $\mathbb{Z}_4$ . The other is to evaluate the thermal exponent of the gauge system at the deconfining temperature. Either method gives a value of  $M_1/M$  which nicely agrees with the ratio  $\sigma_2/\sigma$  evaluated at  $T = 0$ .

### 1.1 The (2+1)D $\mathbb{Z}_4$ gauge model and its dual reformulation

The most general form of  $\mathbb{Z}_4$  lattice gauge model admits two independent coupling constants, with partition function

$$\mathbb{Z}(\beta_f, \beta_{ff}) = \prod_l \sum_{\xi_l = \pm 1, \pm i} e^{\sum_p (\beta_f \mathcal{U}_p + \beta_{ff} \mathcal{U}_p^2 / 2 + \text{c.c.})} , \quad (1.4)$$

in which the gauge field  $U_l$  on the links of a cubic lattice is valued among the fourth roots of the unity and the sum in the exponent is taken over the elementary plaquettes of the lattice. Such a theory can be reformulated as two coupled  $\mathbb{Z}_2$  gauge systems:

$$\mathbb{Z}(\beta_f, \beta_{ff}) = \prod_l \sum_{\{U_l = \pm 1, V_l = \pm 1\}} e^{\sum_p [\beta_f (U_p + V_p) + \beta_{ff} U_p V_p]} , \quad (U_p = \prod_{l \in p} U_l ; \quad V_p = \prod_{l \in p} V_l) . \quad (1.5)$$

From a computational point of view it is useful to exploit a duality relation, switching to a spin model, that in this case is a 3D AT model, expressed as a double Ising spin field plus a coupling term between the Ising variables  $\{\sigma\}$  and  $\{\tau\}$ :

$$S_{AT}(\alpha, \beta) = - \sum_{\langle xy \rangle} [\beta (\sigma_x \sigma_y + \tau_x \tau_y) + \alpha (\sigma_x \sigma_y \tau_x \tau_y)] ; \quad (1.6)$$

the duality is implemented by

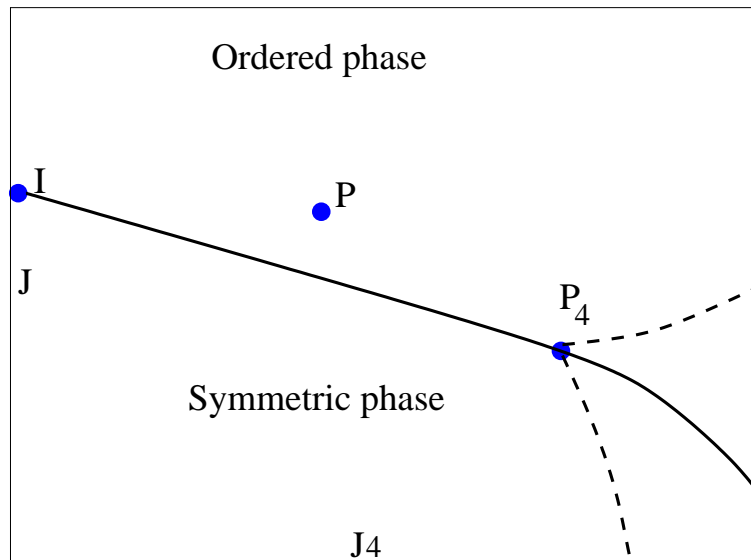
$$\alpha = \frac{1}{4} \ln \left[ \frac{(\coth \beta_f + \tanh \beta_f \tanh \beta_{ff})(\coth \beta_f + \tanh \beta_f \coth \beta_{ff})}{2 + \tanh \beta_{ff} + \coth \beta_{ff}} \right] , \quad (1.7)$$

$$\beta = \frac{1}{4} \ln \left[ \frac{1 + \tanh^2 \beta_f \tanh \beta_{ff}}{\tanh^2 \beta_f + \tanh \beta_{ff}} \right] . \quad (1.8)$$

The choice of working in the dual spin version of the gauge system is strongly motivated by the availability of highly efficient nonlocal Monte Carlo algorithms, in which, moreover, any kind of gauge-invariant observable can be directly embedded in the update procedure.

The  $\mathbb{Z}_4$  gauge model admits, in addition to the fundamental string, a  $k$ -string with  $k = 2$ , corresponding to taking the sources in the double-fundamental representation  $f \otimes f$ .

The phase diagram of this 3D model at  $T = 0$  has been studied long ago [20, 21]. The deconfinement transition, which is weakly first order in the region we are interested in, becomes second order at finite  $T$ , therefore, according to SY conjecture, is described by the order-disorder transition of a 2D AT model. As anticipated in the Introduction, such a transition forms a whole 1-dimensional manifold of critical points (see Figure 1). Along this line, the critical indices (and other universal quantities as well) vary continuously, the endpoints representing a decoupled double Ising system and the 4-state Potts model. The choice of the point in the phase space in which to work is therefore a crucial issue.



**Figure 1:** Sketch of the phase diagram of the Ashkin-Teller model in two dimensions (see Eq. 2.1 for the meaning of the constants  $J$ ,  $J_4$ ). The point labelled  $I$  corresponds to a pair of decoupled critical Ising systems ( $J_4 = 0$ ,  $J = J_c^{\text{Ising}}$ ), while the  $P_4$  point ( $J_4 = J = J_c^{\text{Potts}}$ ) represents the four-state Potts critical point. The solid (self-dual) line between them is the one-dimensional critical manifold along which the critical indices vary with continuity. Beyond  $P_4$ , the self-dual line is no longer critical, and another phase appears (bounded by the two dashed lines departing from  $P_4$ ). The point  $P$  indicates the image of the confined three-dimensional gauge system near  $T_c$  according to the Svetitsky-Yaffe conjecture.

From the data in [17], obtained by means of finite-temperature measurements of Polyakov-Polyakov correlation functions, and particularly from those referring to the point  $P$  identified by  $(\alpha, \beta) = (0.050, 0.207)$ , the string tensions  $\sigma$  and  $\sigma_2$  can be evaluated in

the  $T \rightarrow 0$  limit as temperature-independent quantities:

$$\begin{aligned}\sigma a^2 &= 0.02085(10) , \\ \sigma_2 a^2 &= 0.03356(22) ,\end{aligned}\tag{1.9}$$

where  $a$  is the lattice spacing. Their ratio, which has been argued to equate the central charge of the CFT related to the 2-string, is then given by

$$\frac{\sigma_2}{\sigma} = 1.610(13) .\tag{1.10}$$

## 2. The Svetitsky-Yaffe conjecture and the Sine-Gordon model

The mapping induced by the Svetitsky-Yaffe conjecture leads to a substantial simplification in the study of the critical properties of the deconfining transition, allowing to study it as a standard symmetry-breaking transition which takes place in a spin model. In the present case we deal with the symmetric Ashkin-Teller model in two-dimensions.

The action for this model has the same form of Eq. (1.6), but to clarify the fact that this is the 2-dimensional AT system reached via SY conjecture (as opposed to the three-dimensional one dual to the original gauge theory), we will relabel the spin fields and coupling constants with other names within this context:

$$\mathcal{S}_{AT} = - \sum_{\langle xy \rangle} [J(\sigma_x^1 \sigma_y^1 + \sigma_x^2 \sigma_y^2) + J_4(\sigma_x^1 \sigma_y^1 \sigma_x^2 \sigma_y^2)] .\tag{2.1}$$

Such a model has been extensively studied in the past, and a number of analytic and numerical results have been discovered [20, 22]. Its phase diagram is exactly known (Fig. 1).

The critical line is characterised by the fact that it is self-dual and separates a disordered phase from an ordered one. Since the symmetry which is spontaneously broken by crossing it is the global  $\mathbb{Z}_4$  symmetry of the model, this is also the critical manifold corresponding to the deconfining phase transition of the gauge model. Hence in the following we will concentrate on (a part of) this critical line.

Important analytic results related to the critical line have been worked out by a direct solution of the lattice model [22]. Another interesting approach was carried on in the paper [23], by considering the  $c = 1$  conformal field theory describing the critical line, i.e. the Gaussian model. In such a context it was possible to establish an exact correspondence between lattice/continuum operators and to compute exactly their conformal dimensions.

The advantage of working in the field theoretical setting is that one has the possibility to study the off-critical behaviour near the critical line in a very natural way. This is of great convenience for us, since we will be ultimately interested in the mass spectrum of the theory in the high temperature phase.

Let us briefly sketch how to obtain a field theoretic description of the scaling region near the critical self-dual line of the model. Since the correlation length remains larger than the lattice spacing, the model can be described by the following QFT:

$$\mathcal{A}_{\text{AT}} = \mathcal{A}_{\text{IM}}^{(1)} + \mathcal{A}_{\text{IM}}^{(2)} + \tau \int d^2x (\epsilon_1(x) + \epsilon_2(x)) + \rho \int d^2x \epsilon_1(x) \epsilon_2(x), \quad (2.2)$$

where the meaning of such an expression is quite transparent when compared to the Hamiltonian of the lattice model. The first two terms stand for the conformal field theories with  $c = 1/2$  describing the critical behaviour of the two Ising models, and the latter two terms are respectively the relevant thermal perturbation ( $\epsilon_i(x)$  are the energy operators in the two copies of the Ising model), and the marginal one which moves the system along the critical line. It is also evident by comparison that the couplings  $\tau$  and  $\rho$  are substantially a reformulation of the former  $J$  and  $J_4$  respectively.

In other words, when  $\tau = 0$ ,  $\rho \neq 0$ , the critical line is described by a compactified free massless boson (Gaussian model). A full analysis of the critical line of the Ashkin-Teller model by means of the Gaussian model was established in [23].

The general Hamiltonian with  $\tau \neq 0$  describes the model outside the critical line. It is useful to note that it can be seen as a perturbation of the Gaussian model, and in such a bosonic language the thermal perturbation can be written as  $\cos \beta \varphi$ , where  $\beta$  is a marginal parameter equivalent to  $\rho$ . Hence we are left with

$$\mathcal{A}_{\text{AT}} = \int d^2x \left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \tau \cos \beta \varphi \right), \quad (2.3)$$

which is the action of the Sine-Gordon model. In this notation, we have the high-temperature phase for  $\tau > 0$ , and the low-temperature one for  $\tau < 0$ . It was also shown that the previous model describes the Ashkin-Teller model in the range  $2\pi \leq \beta^2 \leq 6\pi$ . Actually we are interested in the narrower range  $2\pi \leq \beta^2 \leq 4\pi$  from the critical 4-state Potts model to two decoupled critical Ising models. Furthermore, since the confined phase of the gauge theory is mapped in the high-T phase of the Ashkin-Teller model, we will only consider the case  $\tau > 0$ .

Such a QFT is of particular interest because it is integrable, and this is the main reason for rewriting the action of the model near the critical point in a bosonic form<sup>1</sup>.

Integrability means that an infinite number of integrals of motion exists. The main consequence in (1+1)-dimensions is the fact that the scattering theory is very constrained, because the  $S$ -matrix is factorised in products of two-body interactions, and inelastic processes are forbidden.

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<sup>1</sup>It is always possible to *fermionise* the action of the Sine-Gordon model in order to obtain an integrable fermionic theory with the same scattering matrix, namely the massive Thirring model.

These facts allow to write down the so-called Yang-Baxter equations for the 2-particle  $S$ -matrix. Then, such an  $S$ -matrix can be computed exactly by imposing the previous equations and the usual requirements of unitarity and crossing (for a review about Integrable QFTs see [24]).

An obvious consequence is that also the spectrum of the masses of the bound states of the theory is known exactly, since they are represented by the simple poles of the  $S$ -matrix in the physical strip.

It is worth to recall that an important consequence of the Svetitsky-Yaffe conjecture is that the ratio of a given string tension over the temperature of the gauge theory is mapped, near  $T_c$ , onto a corresponding mass of the spin model spectrum. Then, the ratio of string tensions  $\sigma_2/\sigma$  is mapped onto a suitable ratio of masses of the Sine-Gordon model which is known exactly as a function of universal quantities.

In the following we will make some quantitative considerations about the qualitative picture given above in order to make some predictions which will be useful in the context of the gauge theory. Since a detailed analysis of the properties of the scattering theory of the Sine-Gordon model in the context of the Ashkin-Teller model has been done in [25], we will refer to those papers for the details.

## 2.1 Operators correspondence, mass spectrum and correlation functions

The last ingredient we need before exploiting the map to the Sine-Gordon model at its best is the correspondence between the Polyakov loops in higher representation and the operators of the Ashkin-Teller model.

We already know from the Svetitsky-Yaffe original work that the Polyakov loop in the fundamental representation corresponds to the spin operator. Then, following the same reasoning used in [5], it is possible to deduce that the Polyakov loop in the double fundamental representation is related to the so-called *polarisation* operator  $\mathcal{P} = \sigma^1 \sigma^2$ , where  $\sigma^1$  and  $\sigma^2$  are the spin variables defined in (2.1). Its bosonic form and the corresponding anomalous dimensions are given by

$$\mathcal{P} = \sin \frac{\beta}{2} \varphi, \quad X_{\mathcal{P}} = \frac{\beta^2}{8\pi}; \quad (2.4)$$

we also notice that  $\langle \mathcal{P} \rangle = 0$  in the high-T phase of the model.

Sine-Gordon mass spectrum [26]: The exact knowledge of the  $S$ -matrix allows to access to the exact mass spectrum of the theory. Without entering into the details, the spectrum of the SG model is given by a soliton/anti-soliton doublet of fundamental particles of mass  $M$ , and a number of soliton/anti-soliton bound states, called breathers  $B_n$ , whose number



is a function of  $\beta^2$ . By defining the coupling constant  $\xi$  in the following way

$$\xi = \frac{\pi \beta^2}{8\pi - \beta^2} , \quad (2.5)$$

we have that for  $\xi \geq \pi$ , i.e.  $\beta^2 \geq 4\pi$ , no bound states are present and hence the spectrum is given by the soliton/anti-soliton doublet only (repulsive regime).

For  $\xi < \pi$ , i.e.  $\beta^2 < 4\pi$ , we are in the attractive regime and the breathers  $B_n$  appear as simple poles of the S-matrix. Their number and masses are given by the following formula

$$M_n = 2M \sin \frac{n}{2} \xi , \quad 1 \leq n < \left\lceil \frac{\xi}{\pi} \right\rceil . \quad (2.6)$$

Since we are interested in the range  $2\pi \leq \beta^2 \leq 4\pi$ , we immediately realise that, outside the two decoupled Ising point at  $\beta^2 = 4\pi$ , we always have at least one breather of mass  $M_1$  (for  $2\pi \leq \beta^2 \leq 8/3\pi$  we also have the breather  $M_2$  which is however irrelevant for our analysis).

The next step is to associate particle states to operators in the high temperature phase. It has been done in [25] by taking into account their properties of symmetry and locality. The result is that the spin operator only couples to particle states with topological charge equal to one, i.e. involving an odd number of solitons (or antisolitons), and couples to even-labelled breathers only. On the contrary the polarisation operator couples to neutral particle states only, i.e. states with the same number of solitons/antisolitons and odd-labelled breathers.

As a consequence the spin operator is naturally associated to the mass of the soliton, and the polarisation operator is associated to the mass  $M_1$  of the breather  $B_1$ . This means that the string tension of the Polyakov loop in the fundamental representation corresponds to the mass of the soliton, and the the string tension of the double fundamental corresponds to the first breather. Hence, following the Svetitsky-Yaffe conjecture, the ratio of string tensions in the confining phase near the transition is given by

$$\frac{M_1}{M} = 2 \sin \frac{\xi}{2} . \quad (2.7)$$

This result, being a dimensionless ratio, is expected to be universal in the limit  $\tau \rightarrow 0$ . This fact can be explicitly seen by expressing the coupling  $\xi$  in terms of some critical exponent. Not surprisingly, it is indeed possible because the theory is solved also at the critical point in terms of the Gaussian model. By comparing the power-like behaviour of the energy-energy correlator of the Gaussian model:

$$\langle \epsilon(x) \epsilon(0) \rangle_{\text{Gaussian}} \propto \frac{1}{|x|^{\frac{\beta^2}{2\pi}}} , \quad (2.8)$$

with that expected from scaling theory,

$$\langle \epsilon(x) \epsilon(0) \rangle_{\text{Gaussian}} \propto \frac{1}{|x|^{2(d-\frac{1}{\nu})}} , \quad (2.9)$$

it is possible to work out the following relation between  $\xi$  and the thermal critical exponent  $\nu$  (we have  $d = 2$ ):

$$\xi = \pi (2\nu - 1) . \quad (2.10)$$

It yields

$$\frac{M_1}{M} = 2 \sin \frac{\pi}{2} (2\nu - 1) . \quad (2.11)$$

Such a result is very important because it gives an exact prediction for the ratio  $\sigma_2(T)/\sigma(T)$  near the deconfining point at  $T_c$  as a function of the critical exponent  $\nu$ .

Large distance behaviour of correlators: The previous analysis of the mass spectrum allows to compute the leading behaviour of the correlators  $\langle \sigma \sigma \rangle$  and  $\langle \mathcal{P} \mathcal{P} \rangle$  at large distance by means of their spectral expansion over form factors (the interested reader can refer to [27] for the details). On general grounds, the leading behaviour at large distance is expected to obey an exponential decay involving the mass of the lightest state allowed by symmetry and locality.

The analysis of the previous section immediately allows to write down the leading term for  $\langle \sigma \sigma \rangle$  and  $\langle \mathcal{P} \mathcal{P} \rangle$  correlators in the high-T phase of the theory, up to a proportionality constant

$$\begin{aligned} \langle \sigma(x) \sigma(0) \rangle &\sim K_0(M|x|) , & |x| \rightarrow \infty ; \\ \langle \mathcal{P}(x) \mathcal{P}(0) \rangle &\sim K_0(M_1|x|) , & |x| \rightarrow \infty , \end{aligned} \quad (2.12)$$

where  $K_0$  denotes the modified Bessel function of order zero, and  $M, M_1$  are the masses of the soliton and the first breather respectively. It is interesting to notice that the spectral expansion gives the exact asymptotic form of the correlator, and not a generic exponential decay.

An important consequence of this observation is that in the regime in which the Svetitsky-Yaffe correspondence holds we expect the same large distance behaviour for the effective string correction (i.e. the term  $\frac{1}{2} \log R$ ) for both the fundamental and the excited string.

Let us explain this point in more detail. The effective string correction in the case of the cylindric geometry of the Polyakov loop correlators has two very different regimes: for distances  $R$  between the Polyakov loops smaller than  $L/2$  (where  $L$  denotes the length of the lattice in the compactified time direction) the correction is the usual Lüscher term

proportional to  $1/R$ . On the contrary for  $R > L/2$ , it is given by an universal logarithmic correction:  $\frac{1}{2} \log(\frac{2R}{L})$  (see for instance eq. (10) of ref. [28]).

This is the regime (small  $L$ , i.e. high  $T$ ) in which we may expect the dimensional reduction picture to hold, and, according to the identification between 2D spin model and 3D gauge theory observables discussed above, this implies that the prefactor in front of the exponential decay of the two point correlators must be  $1/\sqrt{R}$ . This is exactly the prefactor of the  $K_0$  function and this coincidence represents a non trivial test of the reliability of the dimensional reduction program (see the discussion in Sect. 2.2 of [29]).

What is remarkable in the result of Eq. (2.12) is that this same large distance effective string correction holds unchanged both for the fundamental and for the excited string. This represents a strong constraint for any consistent effective string model for excited k-strings and is one of the exact predictions on the k-string behaviour that we can extract from our dimensional reduction analysis.

Let us summarise the results we discussed in this section in the perspective of applying them to the evaluation of the ratio of string tensions near the critical line:

1. The exact large distance asymptotic behaviour of the correlators of  $\sigma$  and  $\mathcal{P}$  can be considered as a reliable tool to extract the lightest mass which governs their exponential decay. One can separately compute the Polyakov-Polyakov correlators in the representations  $f$  and  $f \otimes f$  with a Monte Carlo simulation and then fit the data in order to extract  $M$  and  $M_1$ . Their ratio  $M_1/M$  is an estimate of the ratio of the string tensions  $\sigma_2/\sigma$  near the deconfining transition, and can be directly compared with its estimate at zero temperature in order to confirm or reject our conjecture.
2. An independent way to compute the ratio  $M_1/M$  is to explicitly use the mass formula as a function of the thermal exponent  $\nu$  of the gauge theory which, according to the SY conjecture, coincides with that of the corresponding 2D model. In particular one can study the finite size behaviour of the plaquette operator (or the susceptibility) and extract the corresponding value of  $\nu$ . Then by plugging it in the mass formula (2.11) one gets another independent estimate of  $M_1/M$ , and again it can be compared to the corresponding estimate of  $\sigma_2/\sigma$  at zero temperature. This is also a direct check that the ratio of string tensions follows the proposed analytic formula.
3. In the large distance regime  $L < 2R$  both the fundamental and the excited string should be affected by the same effective string correction:  $\frac{1}{2} \log R$ .

## 2.2 Baryon vertices and mass spectrum

The general principle invoked in [17] to derive Eq. (1.1) is simply that in a  $d$ -dimensional gauge theory whatever correlation function made with Polyakov loops in the fundamental representation should be described, at sufficiently low temperature and in the IR limit, by a two-dimensional conformal field theory with central charge  $c = d - 2$ .

A simple consequence of this general principle is that, as long as the temperature is far from the critical one, the shape of the world-sheet spanned by the baryon vertices should be temperature independent; this ensures that the baryon static potential has the expected asymptotic form [17]. Depending on the location of external sources some fundamental strings contributing to the baryon vertex may coalesce, giving rise to the  $k$ -string formation. As noticed in [17], the balance of the string tensions for a given vertex gives the following expression for the angles at the center of the junction of three arbitrary  $k$ -strings

$$\cos \vartheta_i = \frac{\sigma_j^2(T) + \sigma_k^2(T) - \sigma_i^2(T)}{2\sigma_j(T)\sigma_k(T)}, \quad \text{and cyclic permutations of the indices.} \quad (2.13)$$

The rigidity of the geometry of the vertex is then ensured by requiring that such angles are kept fixed when the temperature varies. As a consequence, all the string tension ratios are constant up to a given order in  $T$ , namely as far as the effective string picture is valid. In other words, the previous geometrical construction is likely to break down when the system approaches the deconfining temperature, as the string begins to fluctuate wildly.

We can summarise the above consideration by saying that in the low temperature region the trajectory described in the phase space by the gauge system while varying the temperature  $T$  is a line of constant physics, i.e.  $\sigma_2(T)/\sigma(T)$  is constant.

A similar picture emerges when studying the gauge system near the deconfining transition. In the framework of the Svetitsky-Yaffe conjecture the second order phase transition of the gauge system is described by the critical behaviour of a certain 2D spin model. Then, the off-critical scaling region of the latter can be described by a suitable (1+1)D QFT<sup>2</sup>. Focusing on its scattering properties, it is possible to argue that the S-matrix is characterised by its analytic properties [31]. In particular the two-particle elastic scattering matrix is a multivalued function of the Mandelstam variable  $s$  ( $\theta$  is the rapidity which parametrise momentum and energy)

$$s = m_1^2 + m_2^2 + 2m_1m_2 \cosh(\theta_1 - \theta_2), \quad (2.14)$$

with a Riemann sheet possessing three branch points (it is otherwise meromorphic), called physical sheet. The branch points are at  $(m_1 + m_2)^2$ ,  $(m_1 - m_2)^2$  and  $\infty$ , and the cuts

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<sup>2</sup>For a recent application of this approach to 3D  $\mathbb{Z}_3$  gauge theory see [30].

are located on the real line avoiding the interval  $[(m_1 - m_2)^2, (m_1 + m_2)^2]$ , which is the interval where bound state poles can appear (physical strip). Their masses are eventually given by

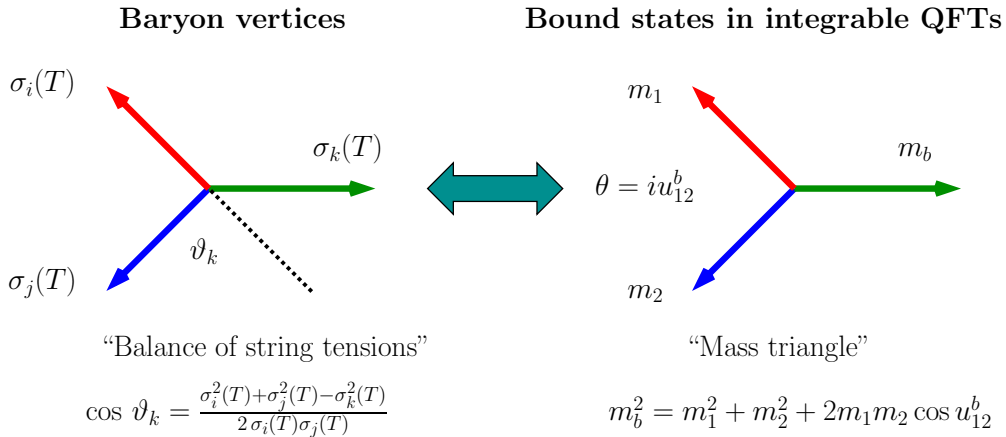
$$m_b^2 = m_1^2 + m_2^2 + 2m_1m_2 \cos u_{12}^b, \quad \text{triangle of masses}, \quad (2.15)$$

where  $\theta = i u_{12}^b$  is the purely imaginary value of the rapidity corresponding to the creation of the particle  $m_b$ .

Let us stress that this is merely a consequence of the kinematics, and it is a generic situation for any (1+1)D QFT. It is nice to see that we are left with the very same structure of angles as Eq. (2.13), which is translated in the so-called “triangle of masses” for the bound state particle  $m_b$ . Interestingly, since such masses play the role of string tensions, we see that they obey the same relation both near  $T = 0$  and near  $T = T_c$ , see Fig. 2.

Coming back to the case analysed in this paper, we can add a further important element to this picture. As explained in the previous Sections, the 3D  $\mathbb{Z}_4$  lattice gauge theory is mapped via Svetitsky-Yaffe to the 2D Ashkin-Teller model near the self-dual critical line, which is in turn described by the Sine-Gordon field theory.

One of the main consequences of the integrability of the latter is the exact knowledge of the mass spectrum. In the present case the process of coalescence of two fundamental strings into a 2-string corresponds to the scattering of a soliton/anti-soliton pair creating the bound state  $B_1$ .



**Figure 2:** Pictorial representation of the relation between baryon vertices and mass spectrum.

For such a process we know that  $u_{SS}^{B_1} = \pi - \xi$  which, once inserted in (2.15), gives

$$M_1^2 = 2M^2(1 - \cos \xi) \quad \rightarrow \quad \frac{M_1}{M} = 2 \sin \frac{\xi}{2}, \quad (2.16)$$

which is nothing but the mass formula used in the previous Section.

The crucial point is now to point out that the balance of string tensions near  $T = 0$  and the mass triangle near  $T_c$  have another common feature. In the former the scaling properties of the baryon potential require that all the angles  $\theta_k$  should not depend on  $T$ , therefore the string tension ratios  $\sigma_k/\sigma$  stay constant as  $T$  varies and define a line of constant physics starting at  $T = 0$ . In the latter the angles involved depend only on the marginal coupling  $\xi$  hence a variation of the relevant coupling  $\tau$  of the 2D model (2.2) generates a line of constant physics starting at  $\tau = 0$ .

In the gauge/CFT map established by the SY conjecture, in order to avoid a mismatch between the RG trajectories of the 3D gauge system and the corresponding 2D model, relevant perturbations of the CFT should correspond to relevant couplings of the critical gauge system. On the gauge side we pass from the  $T \sim 0$  region to the  $T \sim T_c$  by keeping constant the gauge couplings (hence also the lattice spacing) and varying simply the size of the imaginary time direction. Thus the only physical parameter which is varied in passing from low temperature to  $T_c$  is the *reduced temperature*  $t \equiv \frac{T-T_c}{T_c}$  of the gauge system, hence near  $\tau \sim 0$  we have  $\tau = \tau(t)$ , while  $\xi$  is kept constant.

Summing up, the variation of temperature of the gauge system defines a line of constant physics near  $T = 0$  and a similar line near  $T = T_c$ . The numerical work in the next two Sections will show that these two lines are in fact a single one which goes through the whole confining phase.

### 3. Monte Carlo setting and procedure

#### 3.1 Mass ratio by correlators

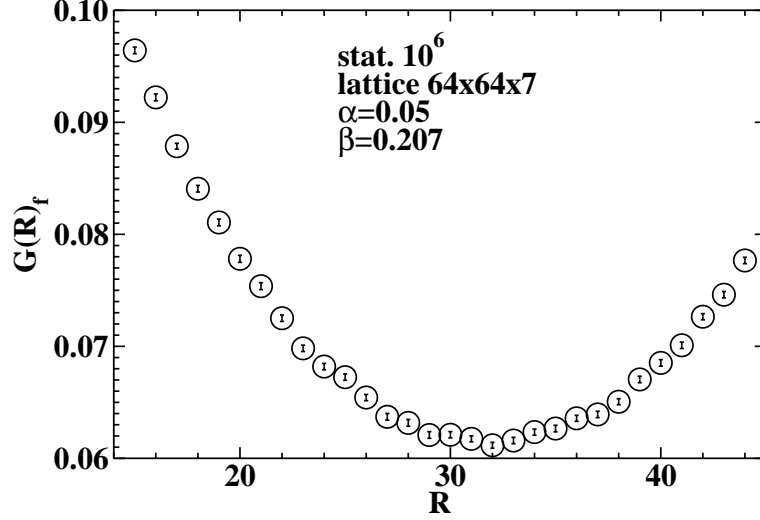
As introduced in Subsection 2.1, we can determine the ratio  $M_1/M$  using the large distance asymptotic behaviour of correlators; actually, exploiting the Svetitsky-Yaffe conjecture, we measured the Polyakov-Polyakov correlators  $G_{\mathcal{R}}(R)$  of the (2+1)D  $\mathbb{Z}_4$  gauge theory:

$$G_{\mathcal{R}}(R) = \langle P_{\mathcal{R}}(0) P_{\mathcal{R}}^{\dagger}(R) \rangle . \quad (3.1)$$

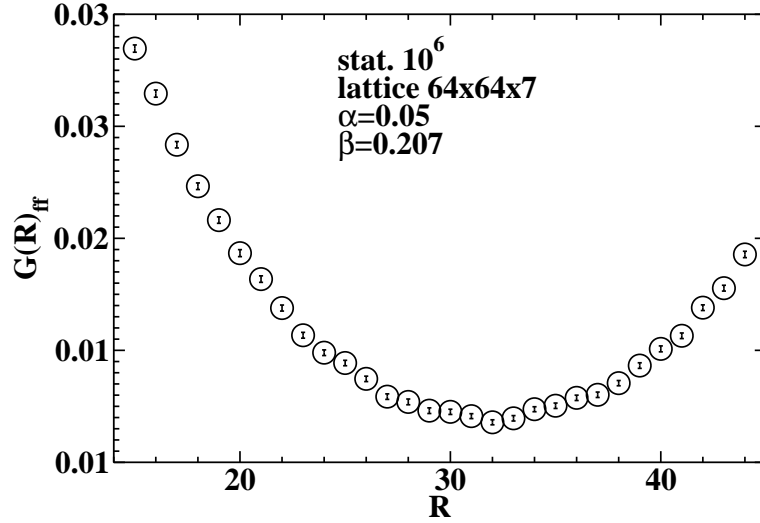
In Section 1.1, we have explained we can study this theory by means of simulations on the dual 3D AT model and in [32] the measurement of Polyakov-Polyakov correlators in both the fundamental and double fundamental representations,  $G(R)_f$  and  $G(R)_{ff}$ , is described in detail.

Note that here, since we are studying the theory near the critical line, the periodic boundary conditions play an important role; in this case there are 16 topologically different surfaces bounded by the two Polyakov lines (4 for each Ising variable, as discussed in [33]), therefore the correlator is the sum of these contributions. In fact, there are only two

important contributions, so we take into account only these two in our simulations (see Figures 3 and 4).



**Figure 3:** Polyakov-Polyakov correlator in the fundamental representation.



**Figure 4:** Polyakov-Polyakov correlator in the double fundamental representation.

We have taken  $10^6$  measures on the  $64^2 \times 7$  lattice in the phase space point  $P$ ;  $N_\tau = 7$  is chosen because it is the lowest possible value above the deconfinement transition. Points on the plots are obtained by independent simulations, one for each value of  $R$  in the range  $[15 \div 44]$ . These data are fitted using an expansion of the  $K_0(mR)$  Bessel function, truncated to first two terms,

$$G(R) = \text{const} \times \frac{e^{-mR}}{\sqrt{mR}} \left[ 1 - \frac{1}{8mR} \right] + \text{“echo terms”}, \quad (3.2)$$

in a range  $[R_{min}, R_{max}]$ , where  $R_{max} = 44$ ; we have verified the results are stable when  $R_{min}$  varies in the range  $[22 \div 33]$ . Therefore, it is possible to determine the two masses:

$$\begin{aligned} a M_{ff} &= 0.0698(15) \quad (\chi^2/\text{d.o.f.} \approx 1.3) , \\ a M_f &= 0.0433(8) \quad (\chi^2/\text{d.o.f.} \approx 1.2) , \end{aligned} \tag{3.3}$$

from which we can determine the ratio:

$$\sigma_2(T \sim T_c)/\sigma(T \sim T_c) = M_{ff}/M_f = 1.612(46) . \tag{3.4}$$

This result, obtained near the critical temperature, is compatible with the zero-temperature value (1.10), providing a strong evidence for our conjecture.

### 3.2 Estimating $\sigma_2/\sigma$ through the thermal exponent $\nu$ with finite-size scaling

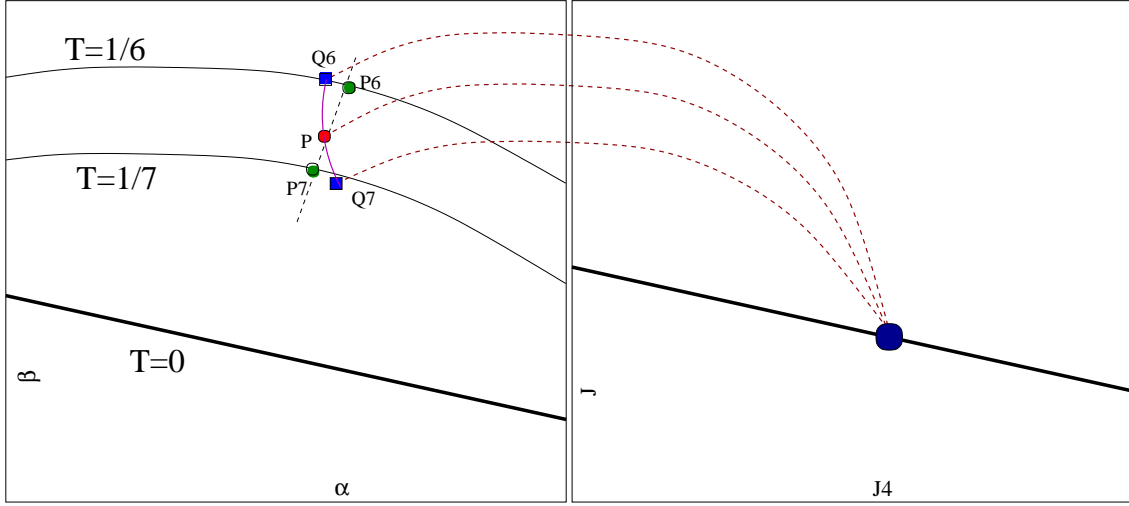
To use the formula for the mass ratio, Eq. (2.11), we need a quite precise estimate for the thermal critical exponent  $\nu$  in the phase space point  $P$  (see Fig. 1). It can be obtained by means of a finite-size scaling analysis once the critical temperature for that choice of couplings has been identified.

The problem is, the system at the coupling  $P$  turns out to be critical for a temperature  $T_c$  such that  $6 < \frac{1}{T_c} < 7$ , hence, having to work with integer inverse temperatures, it is not possible to avoid some approximate method. The idea is then the following: choose a direction in the  $(\alpha, \beta)$  phase space which crosses the critical line, and by moving from  $P$  (in opposite directions) along it find two new points,  $P_7$  and  $P_6$ , at which the system is critical for temperatures  $T = 1/7$  and  $T = 1/6$  respectively. There, perform an estimate for  $\nu$  with a standard finite-size scaling approach. Then, with a linear interpolation, construct the corresponding quantity for the original  $P$  (see Fig. 5).

This first-order approximation, however, is motivated only if the points are close enough that the variation of  $\nu$  is quite small; since the exact shape, in the phase space, of the trajectories of the RG (that is, the set of points which are mapped to the same point on the Gaussian model) is unknown, the best guess is to move from  $P$  along a direction which is perpendicular to the zero-temperature critical line connecting the two decoupled Ising and the 4-state Potts systems.

It has been shown long ago that in the SY context the plaquette operator is mapped into a combination of the unity and the energy operator of the corresponding CFT [34]. Therefore, once the system is made critical, one could extract the thermal exponent  $\nu$  from the finite-size scaling behaviour of the plaquette operator or some related observable that we denote with  $\langle \square \rangle_L$ , where  $L$  is the spacial size of the lattice.





**Figure 5:** In the three-dimensional AT system (on the left), the point  $P$  belongs to a whole line of points which exhibit the same critical behaviour, which is a RG trajectory in phase space (represented as a solid line passing through  $P$ ). The intersections of this line with the finite-temperature critical lines for  $\frac{1}{T} = 6, 7$  are labelled  $Q6$  and  $Q7$ , but their exact position is unknown. Relying on a linear approximation, however, we located the points  $P6, P7$  as described in the text, in order to identify, by interpolation, the point on the dimensionally-reduced AT model (on the right) to which  $P$  is mapped.

In order to exploit the computational advantages of the dual transcription of the gauge model, it is convenient to evaluate directly the internal energy of the 3D AT model defined in (1.6), namely

$$\langle \square \rangle_L \equiv -\frac{1}{3L^2 L_t} \langle S_{AT} \rangle . \quad (3.5)$$

We expect the following finite-size critical behaviour as a function of the spacial side  $L$  of the system:

$$\langle \square \rangle_L = \langle \square \rangle_\infty + b \cdot L^{\frac{1}{\nu}-d} ; \quad (3.6)$$

If one, instead, looks at the corresponding (density of) susceptibility:

$$\langle \chi \rangle_L \equiv \langle (\square - \langle \square \rangle_L)^2 \rangle_L , \quad (3.7)$$

the power-law to compare with has the form:

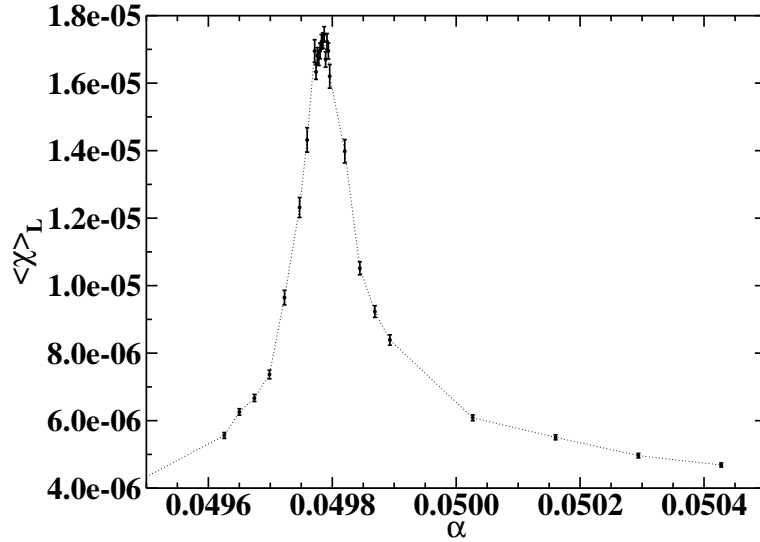
$$\langle \chi \rangle_L = b' \cdot L^{\frac{2}{\nu}-d} , \quad (3.8)$$

with the advantage that no constant additive terms are present, which could largely spoil the stability of the numerical results.

To locate the critical points  $P_6$  and  $P_7$ , we proceeded in the phase space in a dichotomic way along the above-mentioned line from  $P$  and looked for peaks in the plaquette susceptibility on a fixed spatial size system. There is an intrinsic uncertainty on the exact critical values for  $\alpha, \beta$ , but in principle it can be indefinitely shrunk by considering larger and larger lattices.

To perform the simulations, we used a cluster-based nonlocal update algorithm, an adaptation of the Swendsen-Wang prescription, based on alternating global updates on the two Ising subsystems in which the other variables play the role of a frozen background field. The algorithm is described in more detail in [17].

We used  $L = 200$  finite-temperature lattices to find the couplings corresponding to  $P_6$  and  $P_7$ , and for each sampled value of the couplings we took at least  $\mathcal{O}(10^4)$  measurements. By locating the peak in the plaquette susceptibility (see Fig. 6) we could identify the two points with a certain degree of accuracy as  $P_6(0.0500965 \pm 0.0000063, 0.207235 \pm 0.000015)$  and  $P_7(0.0497859 \pm 0.0000077, 0.206478 \pm 0.000019)$ .



**Figure 6:** Behaviour of the plaquette susceptibility in the AT model at  $T = 1/7$  and spatial size  $L = 200$ . The clear peak allowed a precise estimate of the critical point.

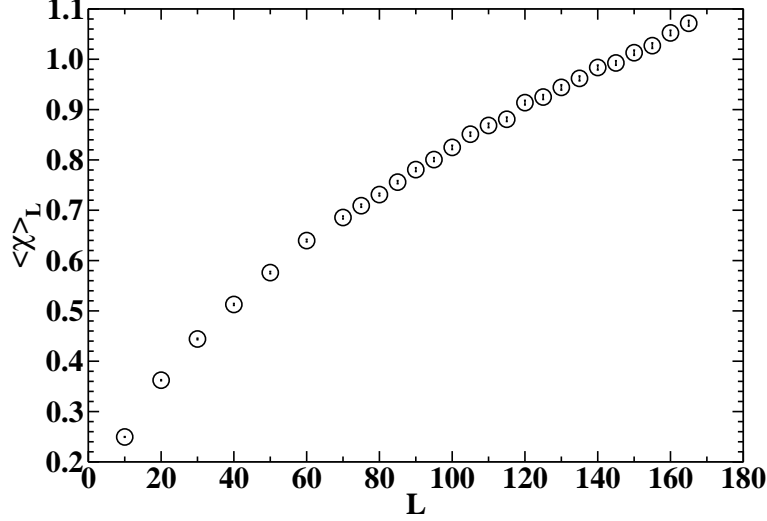
Then, on exactly critical systems at  $P_6$  and  $P_7$ , we took  $\mathcal{O}(10^5)$  measurements of the plaquette at 26 values of spatial side  $L$ , ranging from  $L = 10$  to  $L = 165$ . Not surprisingly, from the bare plaquette data the signal of the power-law behaviour was very noisy due to the presence of a constant background as another fit parameter, so we switched to using the prediction (3.8) for the susceptibility.

The data fitted very well to the expectation from  $L = 70$  already, so we could extract

two values of the critical index  $\nu$  (see Fig. 7):

$$\begin{aligned}\nu_{T=1/6} &= 0.8004(19)[22] ; \\ \nu_{T=1/7} &= 0.7942(18)[38] ,\end{aligned}\tag{3.9}$$

in which the first uncertainty refers to the statistical fluctuations while the second is an estimate of the systematic error in the measurement.



**Figure 7:** Finite-size scaling of the susceptibility of Eq. 3.8 in the AT model at  $T = 1/7$  as a function of the system spatial size. The data allowed a precise estimate of the critical index  $\nu$ .

By linear interpolation along the couplings, the value of  $\nu$  and the (coupling-dependent) critical temperature  $T_c$  was calculated for the very point  $P$ . It was found that  $T_c(P) \simeq 0.1502 \simeq 1/(6.655)$ , which (since the value of  $\sigma$  is well known for  $P$ ) gives the universal ratio

$$\frac{T_c}{\sqrt{\sigma}} = 1.0393(12) .\tag{3.10}$$

From the interpolation, we have  $\nu(P) = 0.7984(19)[27]$ . By plugging it into the formula for the mass ratio (2.11), we obtain the following result:

$$\frac{M_1}{M}(P) = 1.6124(71)[102] ,\tag{3.11}$$

which is compatible with the less accurate estimate coming from the quantities in [17] and thus well supports our conjecture.

## 4. Conclusions

In this paper we studied the ratio of the string tensions  $\sigma_2(T)/\sigma(T)$  near the deconfining point  $T_c$  of a 3D  $\mathbb{Z}_4$  gauge model and compared the result with a general formula which is expected to be true near  $T = 0$  for a generic gauge theory in three or four dimensions.

In this particular case we have combined numerical experiments with known exact results of an integrable 2D quantum field theory that belongs, according to the Svetitsky-Yaffe conjecture, to the same universality class of the critical gauge system.

An interesting property of the integrable model is that the mass ratio of the two physical states of the theory, which should equate the string tensions ratio near  $T_c$ , can be expressed as a simple function of the thermal exponent  $\nu$  (see Eq. (2.11)). Therefore we used two different methods to evaluate such a ratio: either a direct evaluation of the string tensions through Polyakov loop correlators near  $T_c$  (see Eq. (3.4)) or through a measure of  $\nu$  (see Eq. (3.11)). Both the estimates give compatible results which nicely agree with the ratio  $\sigma_2/\sigma$  evaluated at  $T = 0$  (see Eq. (1.10)); for a schematic summary, see Table 1. We then conclude that, at least in this model, the k-string tensions ratios do not depend on  $T$ .

$\sigma_2(T)/\sigma(T)$	Temperature	Method
1.610(13)	$T \simeq 0 + O(T^3)$	Eq. (1.10)
1.612(46)	$T \lesssim T_c$	Eq. (3.4)
1.6124(71)[102]	$T \lesssim T_c$	Eq. (3.11)

**Table 1:** Numerical results.

Even if in Section 2.2 we gave a general RG argument to support this assumption in a wider context, we do not dare to extend such a conjecture to a general gauge system, one reason being that if the deconfinement transition is first order, as is the case in most gauge theories, we do not know a sound argument to support it.

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